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THE STABILITY OF THE SYMMETRIC EQUILIBRIUM POSITIONS OF A REVERSIBLE SYSTEM†

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The behaviour of a system which is reversible with respect to a mapping of the phase space in a neighbourhood of equilibrium positions that do not belong to the fixed set of the mapping is investigated.

1. SOME PROPERTIES OF A REVERSIBLE SYSTEM WITH A PARAMETER

LET US consider an autonomous reversible system of differential equations

$$\mathbf{x}^{\cdot} = \mathbf{f}(\mathbf{x}), \quad \mathbf{M}\mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{M}\mathbf{x}) = 0; \quad \mathbf{x} \in \mathbf{R}^n \tag{1.1}$$

where $\mathbf{M}$ is some constant $n \times n$ matrix. Let $\mathbf{x}^0$ be an equilibrium position, $\mathbf{f}(\mathbf{x}^0) = 0$. Then, if

$$\mathbf{M}^k \mathbf{x}^0 = \mathbf{x}^0 \tag{1.2}$$

(k is an odd number), the equations of the perturbed motion in the neighbourhood of $\mathbf{x}^0$ are reversible [1] with the matrix $\mathbf{M}^k$. Otherwise the reversibility property need not be preserved.

In mechanical problems $\mathbf{M}$ is an involution $\mathbf{M}^2 = \mathbf{E}$ [2]. In such cases the canonical form of the matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{Vmatrix} \mathbf{E}_l & \mathbf{O} \\ \mathbf{O} & -\mathbf{E}_m \end{Vmatrix} \quad (l + m = n)$$

($\mathbf{E}_j$ is the identity matrix of order j). Hence it follows that system (1.1) may be written in the form

$$\begin{aligned} &\mathbf{u}^{\cdot} = \mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \mathbf{v}^{\cdot} = \mathbf{V}(\mathbf{u}, \mathbf{v}); \quad \mathbf{u} \in \mathbf{R}^l, \quad \mathbf{v} \in \mathbf{R}^m \\ &\mathbf{U}(\mathbf{u}, -\mathbf{v}) = -\mathbf{U}(\mathbf{u}, \mathbf{v}), \quad \mathbf{V}(\mathbf{u}, -\mathbf{v}) = \mathbf{V}(\mathbf{u}, \mathbf{v}) \end{aligned} \tag{1.3}$$

and the set of fixed points, i.e. those satisfying (1.2), is the hyperplane $\mathbf{v} = 0$. Let

$$\mathbf{U}(\alpha, \beta) = 0, \quad \mathbf{V}(\alpha, \beta) = 0 \quad (\alpha, \beta - \textbf{const})$$

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Then in the neighbourhood of an equilibrium $\mathbf{u}=\alpha$, $\mathbf{v}=\beta$ the equations of perturbed motion are obtained from (1.3) by replacing $\mathbf{u}$ and $\mathbf{v}$ by $\mathbf{u}+\alpha$, $\mathbf{v}+\beta$ (we will use the same notation for the variables)

$$\mathbf{u}^\bullet = \mathbf{U}(\mathbf{u}, \mathbf{v}, \beta), \quad \mathbf{v}^\bullet = \mathbf{V}(\mathbf{u}, \mathbf{v}, \beta) \tag{1.4}$$

If $\beta=0$, this system, like the original system (1.3), is reversible with matrix $\mathbf{M}$. But if $\beta\neq 0$, the result is generally a reversible system with parameter β system (1.4) transforms into itself under the substitution $t\to -t$, $\mathbf{u}\to\mathbf{u}$, $\mathbf{v}\to -\mathbf{v}$, $\beta\to -\beta$.

Together with system (1.4), we will also consider the system

$$\mathbf{u}^\bullet = \mathbf{U}(\mathbf{u}, \mathbf{v}, -\beta), \quad \mathbf{v}^\bullet = \mathbf{V}(\mathbf{u}, \mathbf{v}, -\beta) \tag{1.5}$$

Since

$$\mathbf{U}(\mathbf{u}, \mathbf{v}, -\beta) = -\mathbf{U}(\mathbf{u}, -\mathbf{v}, \beta), \quad \mathbf{V}(\mathbf{u}, \mathbf{v}, -\beta) = \mathbf{V}(\mathbf{u}, -\mathbf{v}, \beta)$$

It follows that to every solution $\mathbf{u}=\varphi(\mathbf{u}^0, \mathbf{v}^0, t)$, $\mathbf{v}=\psi(\mathbf{u}^0, \mathbf{v}^0, t)$ ($\mathbf{u}^0$ and $\mathbf{v}^0$ being initial values) of Eqs (1.4) there corresponds a solution $\mathbf{u}=\varphi(\mathbf{u}^0, -\mathbf{v}^0, -t)$, $\mathbf{v}=-\psi(\mathbf{u}^0, \mathbf{v}^0, t)$ of system (1.5). This implies the following conclusions.

1. If system (1.4) has a periodic (conditionally periodic) solution at $\beta=\beta^*$, then it must have a periodic (conditionally periodic) solution at $\beta=-\beta^*$.

2. To every invariant set $\mathbf{G}_+$ of system (1.4) at $\beta=\beta^*$ there corresponds an invariant set $\mathbf{G}_-$ at $\beta=-\beta^*$.

3. To every solution of system (1.4) at $\beta=\beta^*$ that is asymptotic to $\mathbf{G}_+$ as $t\to+\infty$ ($t\to-\infty$) there corresponds a solution that is asymptotic to $\mathbf{G}_-$ as $t\to-\infty$ ($t\to+\infty$) at $\beta=-\beta^*$.

4. If $\mathbf{G}_+$ is asymptotically stable (as $t\to+\infty$) then $\mathbf{G}_-$ is unstable (as t increases) and all trajectories go to infinity.

2. REVERSIBLE MECHANICAL SYSTEMS WITH A PARAMETER

The Routh equations of steady motion for a holonomic mechanical system with cyclic coordinates

$$\frac{d}{dt}\frac{\partial \mathbf{R}}{\partial \mathbf{q}^\bullet}-\frac{\partial \mathbf{R}}{\partial \mathbf{q}}=0, \quad \mathbf{R}=\mathbf{R}_2+\mathbf{R}_1+\mathbf{R}_0, \quad \mathbf{R}_1=\sum_{j=1}^{k}\sum_{\alpha=k+1}^{n}\gamma_{\alpha j}(\mathbf{q})\beta_\alpha q_j^\bullet \tag{2.1}$$

($\mathbf{R}_p$ is the form of generalized velocities of order p, and β_α are cyclic constants) are reversible with parameter $\beta=(\beta_{k+1},\ldots,\beta_n)$ these equations are invariant under the substitution $t\to -t$, $q_j\to q_j$, $q_j^\bullet\to -q_j^\bullet$ ($j=1,\ldots,k$), $\beta_\alpha\to-\beta_\alpha$ ($\alpha=k+1,\ldots,n$).

Similarly, the Hamilton equations which follow are reversible with a parameter β that can take values +1 and −1

$$\mathbf{q}^\bullet = \partial\mathbf{H}/\partial\mathbf{p}, \quad \mathbf{p}^\bullet = -\partial\mathbf{H}/\partial\mathbf{q}; \quad \mathbf{H}=\mathbf{H}_2+\beta\mathbf{H}_1+\mathbf{H}_0 \tag{2.2}$$

where $\mathbf{H}_j$ is the jth form of the momenta, and $\mathbf{H}$ is independent of time.

Noting now that systems (2.1) and (2.2) may be stable only over the entire time axis (see, e.g. [3]) and recalling the conclusions of Sec. 1, we obtain the following theorem.

Theorem 1. The system $\mathbf{R}=\mathbf{R}_2+\mathbf{R}_1+\mathbf{R}_0$ ($\mathbf{H}=\mathbf{H}_2+\mathbf{H}_1+\mathbf{H}_0$) is stable (unstable) together with the system $\mathbf{R}=\mathbf{R}_2-\mathbf{R}_1+\mathbf{R}_0$ ($\mathbf{H}=\mathbf{H}_2-\mathbf{H}_1+\mathbf{H}_0$).

In the neighbourhood of the manifold of steady paths of a non-holonomic system, the equations may be taken as follows [4]:

$$
\begin{aligned}
&\mathbf{x}^\bullet = \mathbf{X}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{c}), \quad \mathbf{y}^\bullet = \mathbf{Y}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{c}), \quad \mathbf{w}^\bullet = \mathbf{W}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{c})\\
&\mathbf{z}^\bullet = \mathbf{Z}(\mathbf{x}, \mathbf{y}, \mathbf{w}, \mathbf{z}, \mathbf{c}); \quad \mathbf{x} \in \mathbf{R}^p; \quad \mathbf{y} \in \mathbf{R}^l, \quad \mathbf{w}, \mathbf{z} \in \mathbf{R}^k\\
&\mathbf{X}(\mathbf{x}, \mathbf{y}, 0, 0, \mathbf{c}) \equiv 0, \quad \mathbf{Y}(\mathbf{x}, \mathbf{y}, 0, 0, \mathbf{c}) \equiv 0, \quad \mathbf{W}(\mathbf{x}, \mathbf{y}, 0, 0, \mathbf{c}) \equiv 0\\
&\mathbf{Z}(\mathbf{x}, \mathbf{y}, 0, 0, \mathbf{c}) \equiv 0
\end{aligned}
\tag{2.3}
$$

($\mathbf{c}$ being a parameter); in this form the equations are invariant under the substitution $t \to -t$, $\mathbf{x} \to -\mathbf{x}$, $\mathbf{y} \to \mathbf{y}$, $\mathbf{w} \to \mathbf{w}$, $\mathbf{z} \to -\mathbf{z}$, $\mathbf{c} \to -\mathbf{c}$. Therefore, if dissipative forces are in action at $\mathbf{c} = \mathbf{c}^*$, the equilibrium position $\mathbf{x} = 0$, $\mathbf{y} = 0$, $\mathbf{w} = \mathbf{z} = 0$ is asymptotically stable with respect to $\mathbf{w}$, $\mathbf{z}$ and each of the perturbed motions asymptotically approaches one of the steady motions of the family $\mathbf{x} = \text{const}$, $\mathbf{y} = \text{const}$, $\mathbf{w} = \mathbf{z} = 0$, then the forces at $\mathbf{c} = -\mathbf{c}^*$ are accelerative and each of the perturbed motions will leave a certain neighbourhood of the family.

Using the Lyapunov–Malkin theorem [5], one can prove this assertion in cases when the problem is solved by a linear system. In fact, the linearized equations

$$
\begin{aligned}
&\mathbf{x}^\bullet = \mathbf{A}_1(\mathbf{c})\mathbf{w} + \mathbf{B}_1(\mathbf{c})\mathbf{z}, \qquad \mathbf{y}^\bullet = \mathbf{A}_2(\mathbf{c})\mathbf{w} + \mathbf{B}_2(\mathbf{c})\mathbf{z}\\
&\mathbf{w}^\bullet = \mathbf{A}_3(\mathbf{c})\mathbf{w} + \mathbf{B}_3(\mathbf{c})\mathbf{z}, \qquad \mathbf{z}^\bullet = \mathbf{A}_4(\mathbf{c})\mathbf{w} + \mathbf{B}_4(\mathbf{c})\mathbf{z}
\end{aligned}
\tag{2.4}
$$

involve the constant matrices $\mathbf{A}_1(\mathbf{c})$, $\mathbf{B}_2(\mathbf{c})$, $\mathbf{B}_3(\mathbf{c})$, $\mathbf{A}_4(\mathbf{c})$, which are even functions of the parameter $\mathbf{c}$, and $\mathbf{B}_1(\mathbf{c})$, $\mathbf{A}_2(\mathbf{c})$, $\mathbf{A}_3(\mathbf{c})$, $\mathbf{B}_4(\mathbf{c})$ which are odd functions of $\mathbf{c}$. The characteristic equation of system (2.4) must have $p+1$ zero roots. The remaining roots $\lambda(\mathbf{c})$ are determined from the equation

$$
\det \begin{Vmatrix} \mathbf{A}_3(\mathbf{c}) - \lambda(\mathbf{c})\mathbf{E}_k & \mathbf{B}_3(\mathbf{c}) \\ \mathbf{A}_4(\mathbf{c}) & \mathbf{B}_4(\mathbf{c}) - \lambda(\mathbf{c})\mathbf{E}_k \end{Vmatrix} = 0
\tag{2.5}
$$

and are odd functions of $\mathbf{c}$. If at $\mathbf{c} = \mathbf{c}^*$ it turns out that $\operatorname{Re}\lambda(\mathbf{c}^*) < 0$ for all the roots (dissipative forces dominate), then at $\mathbf{c} = -\mathbf{c}^*$ we have $\operatorname{Re}\lambda(-c^*) > 0$ for all roots and the accelerative forces predominate.

We note [4] that dissipative forces act in the neighbourhood of the steady motions of a non-holonomic system even if there were no dissipative forces in the initial system (before the neighbourhood of the steady motions was reached) and the system was reversible.

3. DISSIPATION ON THE BOUNDARY OF THE STABLE REGION

Let us assume that the characteristic equation (2.5) has one pair of purely imaginary roots and the other roots have negative real parts. Then in that case system (2.3) may be asymptotically stable with respect to part of the variables $\mathbf{w}$, $\mathbf{z}$.

Under these assumptions system (2.3) may be written as follows [6]:

$$
\begin{aligned}
&x_s^\bullet = a_s\rho^2 + X_s(\mathbf{x}, \rho, \theta, \eta) \quad (s = 1, \ldots, \kappa = p + l)\\
&\rho^\bullet = \sum_{j=1}^{\kappa} b_j x_j \rho + R(\mathbf{x}, \rho, \theta, \eta)\\
&\theta^\bullet = \ldots\\
&\eta_i^\bullet = \sum_{j=1}^{q} p_{ij}\eta_j + H_i(\mathbf{x}, \rho, \theta, \eta) \quad (i = 1, \ldots, q = 2k - 2)
\end{aligned}
\tag{3.1}
$$

where a_s, b_j, p_{ij} are constants, all the eigenvalues of the matrix $\|p_{ij}\|$ have negative real parts, and $\mathbf{X}$, R and $\mathbf{H}$ are 2π-periodic functions of θ which vanish identically as functions of $\mathbf{x}$ and θ at $\rho=0$, $\eta=0$; ρ and θ may be interpreted as polar coordinates. In addition, one can always ensure [6] that the functions $\mathbf{X}_s$ contain no terms linear in ρ and η.

The following cases may occur here.

1. A pair of coefficients a_s, b_s exists such that $a_s b_s > 0$. In that case system (3.1) is unstable in Lyapunov's sense. The proof is carried out by standard means, by constructing a Chetayev function in the neighbourhood of the increasing ray of the first non-linear approximation.

2. For all indices $a_s b_s < 0$. Then, using the substitution $\sqrt{(-b_s a_s)}x_s \to x_s$ $(s=1, \ldots, \kappa)$ one can always ensure that the conditions $a_s = -b_s$ $(s=1, \ldots, \kappa)$ are satisfied. Assuming that this is indeed the case, we can now transform to general spherical coordinates

$$
\begin{aligned}
&x_1 = r\cos\varphi_1, \quad x_2 = r\sin\varphi_2\cos\varphi_1, \ldots, x_{\kappa-1} = r\sin\varphi_\kappa\cos\varphi_{\kappa-1}\ldots\cos\varphi_1 \\
&x_\kappa = r\cos\varphi_\kappa\cos\varphi_{\kappa-1}\ldots\cos\varphi_1, \quad \rho = r\sin\varphi_1
\end{aligned}
\tag{3.2}
$$

Then

$$
\begin{aligned}
&r^\bullet = \cos\varphi_1(X_1^* + \sin\varphi_2 X_2^* + \ldots + \sin\varphi_\kappa\cos\varphi_{\kappa-1}\ldots\cos\varphi_2 X_{\kappa-1}^* + \\
&+ \cos\varphi_\kappa\cos\varphi_{\kappa-1}\ldots\cos\varphi_2 X_\kappa^*) + \sin\varphi_1 R^* \\
&r\sin\varphi_1\varphi_1^\bullet = -a_1 r^2\sin\varphi_1 - X_1^* + r^\bullet\cos\varphi_1 \\
&\eta_i^\bullet = \sum_{j=1}^{q} p_{ij}\eta_j + H_i^*(r, \varphi, \theta, \eta) \quad (i=1, \ldots, q)
\end{aligned}
\tag{3.3}
$$

where $\mathbf{X}^*$, R^*, H^* are the functions $\mathbf{X}$, R, $\mathbf{H}$ after the substitution (3.2).

Consider the function

$$
V = r\exp(-a_1\cos\varphi_1) + W(\eta), \quad \sum_{s=1}^{q}\frac{\partial W}{\partial\eta_s}\sum_{j=1}^{q} p_{sj}\eta_j = -(\eta_1^2 + \ldots + \eta_q^2)
$$

which is sign-definite relative to the variables r, η. The derivative of V along trajectories of (3.3) is

$$
\begin{aligned}
&V^\bullet = -a_1^2\exp(-a_1\cos\varphi_1)r^2\sin\varphi_1 - (\eta_1^2 + \ldots + \eta_q^2) + \\
&+ \exp(-a_1\cos\varphi_1)\{-a_1 X_1^* + (1 + a_1\cos\varphi_1)[\cos\varphi_1(X_1^* + \sin\varphi_2 X_2^* + \\
&+ \sin\varphi_3\cos\varphi_2 X_3^* + \ldots + \sin\varphi_\kappa\cos\varphi_{\kappa-1}\ldots\cos\varphi_2 X_{\kappa-1}^*\ldots\cos\varphi_2 X_{\kappa-1}^* + \\
&+ \cos\varphi_\kappa\cos\varphi_{\kappa-1}\ldots\cos\varphi_2 X_\kappa^*) + \sin\varphi_1 R^*]\} + \sum_{j=1}^{q}\frac{\partial W}{\partial\eta_j}H_j^*
\end{aligned}
$$

Since the functions X_s contain no terms linear in ρ, η, and $\mathbf{X}$, R, $\mathbf{H}$ vanish at $\rho=0$, $\eta=0$, it follows that

$$
\begin{aligned}
&V^\bullet = -a_1^2\exp(-a_1\cos\varphi_1)r^2\sin^2\varphi_1 - (\eta_1^2 + \ldots + \eta_q^2) + \\
&+ r^2\sin^2\varphi_1\Psi(r, \varphi, \theta, \eta) + r\sin\varphi_1\sum_{j=1}^{q}\eta_j\Psi_j(r, \varphi, \theta, \eta) + \\
&+ \sum_{i=1}^{q}\sum_{j=1}^{q}\Psi_{ij}(r, \varphi, \theta, \eta)\eta_i\eta_j
\end{aligned}
$$

where Ψ, Ψ_j, Ψ_{ij} vanish at $r=0$, $\eta=0$.

The function V just constructed meets all the conditions of Rumyantsev's theorem [7] on stability with respect to the variables r, $\eta_1, \ldots, \eta_q$ and asymptotic stability with respect to $\rho = r\sin\varphi_1$, $\eta_1, \ldots, \eta_q$. Also, it is evident from (3.3) that $\mathbf{x} \to \text{const}$ as $(\rho, \eta) \to 0$.

Theorem 2. If a pair of coefficients a_s, b_s exists such that $a_s b_s > 0$, then the trivial solution $\mathbf{x}=0$, $\rho=0$, $\eta=0$ of system (3.3) is unstable in Lyapunov's sense. But if $a_s b_s < 0$ for all such pairs, then the trivial solution is stable, in fact, asymptotically stable with respect to part of the variables ρ, η, and each of the perturbed motions asymptotically approaches a steady motion of the κ-family.

$$\mathbf{x} = \mathbf{const}, \quad \rho = 0, \quad \eta = 0$$

4. EXAMPLE. PERMANENT ROTATIONS OF A HEAVY CONVEX RIGID BODY ON AN ABSOLUTELY ROUGH FIXED HORIZONTAL PLANE

Retaining the notation of [8], let the equations of motion be

$$\begin{aligned} &\Theta\omega^{\cdot} + \omega \times (\Theta \cdot \omega) = mg\mathbf{r} \times \gamma - m\mathbf{r} \times [\omega^{\cdot} \times \mathbf{r} + \omega \times r^{\cdot} + \omega \times (\omega \times \mathbf{r})] \\ &\gamma^{\cdot} + \omega \times \gamma = 0 \end{aligned} \tag{4.1}$$

To get a closed system, we add a relation $f(\mathbf{r})=0$ defining the surface of the body: the relationship between the vectors $\mathbf{r}$ and γ will be

$$\gamma = -\operatorname{grad} f(\mathbf{r}) / |\operatorname{grad} f(\mathbf{r})|$$

System (4.1) is reversible; it is a special case of (1.3) with vectors $\mathbf{u}=\gamma$, $\mathbf{v}=\omega$. As shown in [8], they have a particular solution

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1, \quad \omega_1 = \omega_2 = 0, \quad \omega_3 = \omega_0 \tag{4.2}$$

(ω_0 is an arbitrary constant), corresponding to permanent rotation of the body at angular velocity ω_0 around one of its principal central axes of inertia, which is the vertical axis, provided that it is orthogonal to the surface of the body. In the neighbourhood of (4.2) the equations of perturbed motion will have the same form as (2.3), with

$$x = \omega_3 - \omega_0, \quad y = \gamma_3 - 1, \quad w = (\gamma_1, \gamma_2)^T, \quad z = (\omega_1, \omega_2)^T$$

Therefore, if the solution (4.2) with rotation in one direction $\omega_0 < 0$ is asymptotically stable with respect to γ_1, γ_2, ω_1, ω_2, then on rotation in the other direction ($\omega_0 > 0$) the motion will be unstable and each trajectory will leave a certain neighbourhood of zero in the space of the variables γ_1, γ_2, ω_1, ω_2,

Sufficient conditions for asymptotic stability to a first approximation may be found in [8]. For rotation of a body with a stable equilibrium position around the principal axis corresponding to the largest moment of inertia, these conditions are

$$\omega_0 < \omega_* < 0$$

If $\omega_0 = \omega_*$, the characteristic equation has a pair of purely imaginary roots and two roots with negative real parts [8], and system (4.1) has a periodic solution (Hopf bifurcation) [8]. It will then follow from Secs 1 and 2 that if $\omega_0 = -\omega_* > 0$ the pair of purely imaginary roots will also produce a periodic motion.

Note that if $\omega_0 = 0$ (equilibrium), the equations of perturbed motion are reversible, the characteristic equation has two pairs of purely imaginary roots and, in the neighbourhood of the equilibrium position, two families of periodic Lyapunov motions exist [2].

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